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# Measures in the geometric quantization of field theories

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**Abstract**

In this article, it is shown that for the standard symplectic form on the space of compactly supported sections of a symplectic fibre bundle, there is no locally-finite Borel measure which is preserved by the Hamiltonian flows of even a quite restricted set of functions on this space. As this means that some of the operators arising in geometric quantization associated to classical observables would not be Hermitean, the result suggests that one should consider quotients by gauge groups as classical phase spaces to avoid this problem.


*Keywords:* Geometric quantization; Measure theory; Symplectic fibre bundles; Frechet manifolds

## 1. Introduction and statement of the results

One of the basic features of Hamiltonian flows on finite-dimensional symplectic manifolds is that they preserve the symplectic form $\omega$ and hence the natural volume form $\omega \wedge \omega \wedge \cdots \wedge \omega$. Now one could try to obtain a similar result on the phase space of a field theory which is often a Frechet manifold of sections of a fibre bundle. Sometimes this fibre bundle is a symplectic fibre bundle, e.g. in the case that one has a Cauchy correspondence

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of first order, i.e. a Frechet diffeomorphism between the space of initial values up to the first derivatives and the space of classical solutions. Then one can construct an associated symplectic form on the space $\Gamma(\pi)$ of sections of the symplectic fibre bundle $\pi$ (which is defined below) and ask whether there is a nonzero locally finite Borel measure on $\Gamma(\pi)$ that is preserved by the group of Hamiltonian flows on $\Gamma(\pi)$. We will show that the answer is no. This implies that in this case, the operators corresponding to quantum observables in geometric quantization are not Hermitean.

We will first describe shortly the framework of geometric quantization. Details can be found in [9].

*Geometric quantization* (a good overview of which is provided by the books of Woodhouse [14] and Sniatycki [13]) is basically a replacement of the abstract Hilbert space of some quantum theory by the space of smooth sections of a complex line bundle $l$ over the space $(\Gamma(\pi))$ of classical solutions which is in general a Frechet manifold of sections of a fibre bundle $\pi : E \to M$. Here we assume for later use that $M$ is equipped with a volume form $d\mathrm{vol}_M$. The Hilbert space operators are then replaced by some linear Hermitean first order differential operators in $\Gamma(l)$. Thus on the level of observables, geometric quantization is a mapping:

$$Q : C^\infty(\Gamma(\pi), \mathbb{R}) \supset S \to \mathrm{End}(\Gamma(l)), \quad Q(f)(\psi) := -i\hbar\nabla_{X_f}\psi + f\psi$$

where $l$ is a Hermitean complex line bundle on $\Gamma(\pi)$ with a Hermitean connection $\nabla$ whose curvature is a (weakly) symplectic two-form $\hbar^{-1}\Omega$ defining Hamiltonian vector fields $X_f$ for some functions $f$ by $\Omega(X_f, \cdot) = \mathrm{d}f(\cdot)$, and $S$ is the subset of all functions with a Hamiltonian vector field (thus $S$ is closed under the Poisson bracket). By $\mathrm{End}(\Gamma(l))$ we mean the set of linear operators acting on $\Gamma(l)$. The operators obtained are first order differential operators, i.e. they are not only elements of $\mathrm{End}(\Gamma(l))$, but also elements of $\mathrm{End}(j^1 l)$, where $j^1 l : J^1 l \to \Gamma(\pi)$ is the first jet bundle of $l$. A condition ensuring the existence of the line bundle above is given by the following theorem (for the proof cf. [14]).

**Theorem 1.** *Let $\mathcal{M}$ be a (possibly infinite-dimensional Frechet) manifold carrying a (weakly) symplectic form $\Omega$. Then there is a Hermitean line bundle with a connection of curvature $\hbar^{-1}\Omega$ if and only if the cohomology class of $\Omega$ in $H^2(\mathcal{M}, \mathbb{R})$ lies in $H^2(\mathcal{M}, \mathbb{Z})$.*

We will call a manifold *prequantizable* if it satisfies this condition.

The map $Q$ satisfies Dirac's famous axiom system for correspondences between classical and quantum observables ([3]):

1. The map $f \to Q(f)$ is $\mathbb{R}$-linear,
2. For $f$ constant, $Q(f)$ is the corresponding multiplication operator,
3. The map $Q$ is an algebra homomorphism, more precisely, the following diagram commutes:

$$\begin{array}{ccc} S \times S & \xrightarrow{\{\cdot,\cdot\}} & S \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathrm{End}(\Gamma(l)) \times \mathrm{End}(\Gamma(l)) & \xrightarrow{-i\hbar[\cdot,\cdot]} & \mathrm{End}(\Gamma(l)) \end{array}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket with respect to the symplectic form $\Omega$, and $[\cdot, \cdot]$ means the commutator of linear operators.

In addition, we need a measure $\mu$ on the classical phase space for which all quantum operators are Hermitean with respect to the $L^2$-norm if restricted to smooth square-integrable sections of $l$ (as in general $\mu$ need not to be finite). Recall that in the light of the Kopenhagen interpretation Hermiticity is important because only in that case all expectation values are real. Now, we will see that there is no such measure. This will be done in Proposition 3.

Let us introduce some non-standard notation. For $n \leq \infty$, let $\tilde{\Gamma}^n(\pi)$ denote the space of all $n$ times continuously differentiable sections of the fibre bundle $\pi : E \to M$, i.e. $\tilde{\Gamma}^n(\pi) := \{\gamma \in C^n(M, E) : \pi \circ \gamma = \mathbf{1}_M\}$. Let $\Gamma^n(\pi)$ be the corresponding spaces of sections of compact support which in the case of a fibre bundle means that the section coincides with a fixed reference section outside a compact set: fix $\gamma_0 \in \tilde{\Gamma}^n(\pi)$, then:

$$\Gamma^n(\pi) = \{s \in \tilde{\Gamma}^n(\pi) | \exists \text{ compact } C \subset M \text{ with } s|_{M\setminus C} = \gamma_0|_{M\setminus C}\}.$$

The spaces $\tilde{\Gamma}^0(\pi)$ and $\Gamma^0(\pi)$ can be equipped with the metric of uniform convergence on compact subspaces, i.e.

$$\mathrm{d}(\gamma_1, \gamma_2) := \sum_{n\in\mathbb{N}} \min\left(\frac{1}{2^n}, \max_{x\in K_n}(\mathrm{d}_0(\gamma_1(x) - \gamma_2(x)))\right). \tag{1}$$

for an increasing sequence $K_n$ of compact sets with $\bigcup_n K_n = M$, where $d_0$ is the Riemannian distance (with respect to an arbitrary Riemannian metric on $E$) in the submanifold $\pi^{-1}(x)$. This generates the *compact-open topology* $\tau$ on $\Gamma^0(\pi)$ a subbasis of which is formed by all sets $(C, O) := \{\gamma \in \Gamma(\pi) : \gamma(C) \subset O\}$ of sections that map a fixed compact set $C \subset M$ into a fixed open set in $E$, analogously for spaces of higher differentiability (note that although this metric and topology is well-defined for general manifolds $M$, $\Gamma^n(\pi)$ will be a Frechet manifold if and only if $M$ is compact!). Set $\sigma_\pi := \sigma(\tau)$, where for a family of subsets $K$ the term $\sigma(K)$ means the smallest $\sigma$-algebra containing $K$. By a Borel measure we mean a measure on $\sigma_\pi$.

Finally, note that $\sigma_\pi = \sigma(\{pr_{\{m\}} | m \in M\}) = \{pr_{\{m\}}(B) | m \in M, B \in \mathcal{B}(\pi^{-1}(\{m\}))\}$, where $pr_{\{m\}} := ev_m : \Gamma(\pi) \to \pi^{-1}(\{m\})$, the evaluation map at $p$, and $\mathcal{B}(\pi^{-1}(\{m\}))$ is the Borel-$\sigma$-algebra of $\pi^{-1}(\{m\})$.

The *symplectic form* used here is quite common in geometric quantization of field theories and goes probably back to Chernoff and Marsden ([2], [10], [4] for the case of a trivial bundle, [14] and [9] for an overview).

A crucial tool of the construction is the identification of a tangent vector $V$ respectively the value of a vector field $V$ on $\tilde{\Gamma}^{k+1}(\pi)$ at a fixed section $\gamma$ with a vector field *along* $\gamma$, i.e.

a section of $\gamma^* T^v E$:

$$\hat{}: T|_\gamma \tilde{\Gamma}^{k+1}(\pi) \to \tilde{\Gamma}^k(\gamma^* T^v E), \qquad \hat{V}|_\gamma(p) := \mathcal{L}_V ev_p$$

or equivalently,

$$\hat{V}|_\gamma : p \mapsto \partial_t(\gamma_t(p)),$$

where $p \in N$, $\gamma_t$ a curve representing $V(\gamma)$. This means, we fix a point $p \in M$ and note the direction in which it is moved infinitesimally by the family of maps $\gamma_t$. If we start with a tangent vector $V$ at $\tilde{\Gamma}^k(\pi)$ tangent to the submanifold $\Gamma^k(\pi)$ of compactly supported sections, then $\hat{V}|_\gamma$ has also compact support because $\bigcup_{t\in[-1,1]} \gamma_t$ has compact support.

The definition of the symplectic form on $\Gamma^1(\pi)$ is relatively simple. To every $(p, 0)$-tensor field $A$ on $E$ we can associate a $(p, 0)$-tensor field $\tilde{A} \in \Gamma((T^*M)^{\otimes p})$ on $\Gamma^1(\pi)$ by the prescription:

$$\tilde{A}(V_1, \cdots, V_p)(\gamma) := \int_M A(\hat{V}_1^\gamma, \cdots, \hat{V}_p^\gamma)\, \mathrm{dvol}_M$$

where each $\hat{V}_i^\gamma$ is the corresponding vector field along $\gamma$. Then, by means of an arbitrary auxiliary Riemannian metric $g$ on the total space $E$, to every $(p, q)$-tensor field $A$ on $E$ we can associate a $(p, q)$-tensor field $\tilde{A}$ on $\Gamma^1(\pi)$ by the prescription

$$\begin{aligned} &\tilde{g}(\tilde{A}(V_1, \cdots, V_p), V_{p+1} \otimes \cdots \otimes V_{p+q})(\gamma) \\ &\quad := \int_M g(A(\hat{V}_1^\gamma, \cdots, \hat{V}_p^\gamma), \hat{V}_{p+1}^\gamma \otimes \cdots \hat{V}_{p+q}^\gamma)\, \mathrm{dvol}_M \end{aligned}$$

This construction shares many good properties such as naturality under isometric embeddings (for details cf. [9]). Moreover, it induces a chain map as follows:

**Theorem 2** ([12],[9]). *The map* $\tilde{}: \Lambda^*(E) \to \Lambda^*(\Gamma(\pi))$ *is a chain map, i.e.* $\mathrm{d} \circ \tilde{} = \tilde{} \circ \mathrm{d}$. *Moreover the kernel of* $\tilde{}$ *are exactly the forms which are zero along the fibres of* $\pi$.

Now we assume $\pi$ to be a symplectic fibre bundle, where the fibre manifold carries a symplectic form $\omega$ smoothly depending on the base point in $M$. Then $\Gamma^1(\pi)$ is equipped with the two-form $\tilde{\omega}$ which assigns to any two tangent vectors $X, Y$ at a section $\gamma$ (with $\hat{X}, \hat{Y}$ the associated vector fields along $\gamma$) the number:

$$\Omega(X, Y) := \tilde{\omega}(X, Y) = \int_M \omega(\hat{X}, \hat{Y})\, \mathrm{dvol}_M, \tag{2}$$

and an almost immediate consequence from the theorem above is that $\Omega$ is a (weakly) symplectic form on $\Gamma^1(\pi)$. For further details of the constructions above cf. [9]. This procedure is motivated by the fact that for a field theory on a bundle $\tau : A \to N$ over a globally hyperbolic Lorentzian manifold $N$, one can often find a Cauchy correspondence of first order, i.e. a Frechet diffeomorphism between the classical solutions and the values of the field up to first order of derivatives on a Cauchy surface $M$ with normal vector field $\nu$, i.e. classical solutions can be identified with sections of the bundle $\pi = \nu^* \, \mathrm{d}\tau : TA \to M$. Often $A$ carries a bilinear form by which $TA$ can be identified with $T^*A$, hence we have sections of a symplectic fibre bundle.

Now we can state the results of this article:

**Proposition 3.** *Let $\pi : E \to M$ be a symplectic fibre bundle. We equip $\Gamma^n(\pi)$ with the weakly symplectic structure $\Omega$ defined by* Eq. (2). *We suppose $(\Gamma^n(\pi), \Omega)$ to be prequantizable. Then there is no nonzero locally finite Borel measure $\mu$ on $\Gamma^k(\pi)$, $1 \leq k \leq \infty$ as subset of $\Gamma^l(\pi)$, $0 \leq l \leq k$ such that all operators $Q(\tilde{f})$, where f is a compactly supported function on E, are Hermitean.*

The proposition will be proved as corollary of the following proposition:

**Proposition 4.** *Let $\pi : E \to M$ be a symplectic fibre bundle. We equip $\Gamma^n(\pi)$ with the weakly symplectic structure defined by* Eq. (2). *There is no nonzero locally finite Borel measure $\mu$ on $\Gamma^k(\pi)$ as subset of $\Gamma^l(\pi)$, $l \leq k \leq \infty$ which is invariant under the Hamiltonian flows of its smooth $L^2$-functions of the form $\tilde{f}$, where f is a compactly supported smooth function on E.*

## 2. Proof of the propositions

The strategy of the proof is the following: we first pick a trivial neighborhood $U$ of the fibre bundle and consider the induced measure on sections over this neighborhood which we consider in a trivialization adjusted to the symplectic structure in an open set $V \subset \pi^{-1}(U)$ (which then can be understood as $U$ times a subset of Euclidean $R^{2n}$ with its standard symplectic form) so that in this trivialization $\tilde{\Gamma}^1(\pi|_V) = C^1(U, V)$. If we try to construct a measure on $C^1(U, V)$ the first idea would be to take an image measure under the canonical embedding $\mathcal{P} : V \hookrightarrow C^1(U, V)$ whose image is the set of point maps which we denote by $P$. On the elements of the subbasis it takes the form $\mu(C, O) = \mathcal{L}(O)$ where $\mathcal{L}$ denotes the Lebesgue volume in $V$, and for every element $(C, O)$ of the subbasis of the compact-open topology defined above. Then we show that in this trivialization the measure would have to be an element of $im(\mathcal{P})$. But if we change locally the trivialization to get another adjusted trivialization, we get another map $\mathcal{P}$ and another measure. This produces the contradiction.

Without restriction of generality we can consider the case $l = 0$, $k = \infty$. This is because we have continuous embeddings of $\Gamma^i(\pi) \to \Gamma^0(\pi)$ such that for every measure on $\Gamma^i(\pi)$ we can consider its image measure on $\Gamma^0(\pi)$.

Without restriction of generality let $M$ be connected. Choose a section $\gamma$ and an open neighborhood of the form $(M, B)$ of $\gamma$ (where $B$ is a neighborhood of the image of $\gamma$) such that $0 < \mu(M, B) < \infty$ and such that the line bundle $l$ is trivial over $(M, B)$. This exists because of local finiteness and as every open set in the compact-open topology contains a ball with respect to the metric defined in Eq. (1). Now pick a trivializing neighborhood

$U \subset M$ of $\pi$ with compact closure $\overline{U}$. Then via the trivialization we have $\pi^{-1}(U) = U \times F$ where the induced symplectic form on this product depends as well on the footpoint in $U$ as on the footpoint in $F$. For every $u \in U$ we choose an open neighborhood of $\gamma(u)$ which has a Darboux trivialization, i.e. a diffeomorphism mapping the neighborhood to an open set in the standard symplectic space $\mathbb{R}^{2n}$. Because of compactness of $\overline{U}$ one can consider this neighborhood $N$ as one and the same subset of $\mathbb{R}^{2n}$ with smoothly varying symplectic structure. Then it is easy to show that one can choose the Darboux trivialization as well smoothly depending on the footpoint $u \in U$ using the following lemma:

**Lemma 5** (smooth Darboux coordinates). *Let $\overline{U}$ be a compact manifold* (*with or without boundary*), *let $\pi : P \to \overline{U}$ be a symplectic fibre bundle*, *then for every section $\gamma$ of $\pi$ there is a neighborhood $V$ of $\gamma(\overline{U})$ which is symplectomorphic to $\overline{U} \times V$*, *where $V$ is an open set in $\mathbb{R}^{2n}$ with its standard symplectic structure*. *Under this symplectomorphism*, *$\gamma(\overline{U})$ corresponds to $\overline{U} \times \{0\}$*.

**Proof** (Proof of the lemma). First note that the notion of symplectomorphism is not quite correct here as the form is defined for vector fields along the fibre and is degenerate when extended by zero on the whole tangent space of $P$. Take a tubular neighborhood $T$ of the image of the section $\gamma$. Then consider the proof of existence of Darboux coordinates as given in [7], pp. 10–11. All constructions used there depend smoothly on the given symplectic form, except the choice of the one-form $\lambda$. As $\mathrm{d}\lambda$ is prescribed and as $T$ is fibrewise simply-connected, the only freedom we have over $u \in \overline{U}$ is adding the differential of a function on $\pi^{-1}(u) \cap T$, i.e. replacing $\lambda \to \lambda + \mathrm{d}f$, with $f$ vanishing at zero without restriction of generality. So for a $\lambda_0$ with $\mathrm{d}\lambda_0 = \omega - \omega_0$ we define,

$$A := \{\lambda_0 + \mathrm{d}f : f(0) = 0\}$$

which is an affine subspace of $\Lambda^1(\pi^{-1}(u) \cap T)$. Then define an arbitrary Riemannian metric $g$ on $T$ and choose $\lambda \in A$ such that it minimizes the functional $S(\lambda) = \int_{\pi^{-1}(u)\cap T} g(\lambda, \lambda)$. This is a convex functional and hence fixes the choice of $\lambda$. As all constructions depend smoothly on the basepoint, so does the resulting diffeomorphism onto open sets in $\mathbb{R}^{2n}$ containing 0. □

Therefore, the restriction $\pi|_{V\cap\pi^{-1}(U)}$ of the fibre bundle $\pi$ is isomorphic to a trivial bundle $U \times V \to U$ via an isomorphism which preserves the symplectic structure. Moreover, we can assume that $U$ and $V$ are simply connected. Then in $V$, the symplectic form $\omega$ is exact, $\omega = \mathrm{d}\theta$ for a one-form $\theta$, and so for $\Theta := \tilde{\theta}$ we have:

$$\mathrm{d}\Theta = \mathrm{d}\tilde{\theta} = \tilde{\mathrm{d}\theta} = \tilde{\omega} = \Omega,$$

thus, $\hbar^{-1}\Theta$ is the connection one-form with respect to this trivialization in the conventions of the Woodhouse book ([14], pp. 155–162, and 265–269). Therefore, by using the trivial derivative in this trivialization we can write:

$$Q(f)(\psi) := -i\hbar X_f(\psi) - \Theta(X_f)\psi + f\psi.$$

Thus, restricted to the trivializing neighborhood the quantum operators in geometric quantization are a sum of multiplication operators and a constant multiple of the trivial derivation along a Hamiltonian vector field $X_f$. Therefore, the requirement of Hermiticity is equivalent to Hermiticity of $X_f$. If $X_f$ has a flow, this is equivalent to $\cdot \circ Fl^t_{X_f} =: S_t$ to be a unitary operator in $L^2(\mu)$ because the flow of $X_f$ is its exponential in the Hilbert space $Hom(L^2(\mu), L^2(\mu)) = L^2(\mu) \otimes L^2(\mu)^*$.

Now,

$$\langle f, g\rangle = \int_{\Gamma(\pi)} f(\gamma)g(\gamma)\,\mathrm{d}\mu(\gamma),$$

$$\langle S_t f, S_t g\rangle = \int_{\Gamma(\pi)} f(Fl^t_{X_f}(\gamma))g(Fl^t_{X_f}(\gamma))\,\mathrm{d}\mu(\gamma) = \int_{\Gamma(\pi)} f(\gamma)g(\gamma)\,\mathrm{d}((Fl^t_{X_f})^*\mu)(\gamma)$$

and unitarity of $S_t$ is equivalent to the invariance of $\mu$ under the flow of Hamiltonian vector fields, $\Phi_t^*\mu = \mu$. Thus Hermiticity is equivalent to the statement that the Lie derivative of the measure along Hamiltonian vector fields of functions on the space of sections vanishes. We take functions of the form $\tilde{f}$ where $f$ is a compactly supported function on the total space of the bundle. Their Hamiltonian vector fields $X_{\tilde{f}} = \tilde{X}_f$ of course possess a local flow (here $X_f$ are vertical vector fields by definition). Thus Proposition 3 is implied by Proposition 4 which we are going to prove now.

From the original measure $\mu$ on $\Gamma^0(\pi)$ we want to construct a finite measure $\mu'$ on $\tilde{\Gamma}^0(\pi|_{V\cap\pi^{-1}(U)})$. This is done by the following definition: choose a sequence of compact sets $K_i \subset K_{i+1} \subset M$ whose union is $M$, then $(M, B) = \bigcap_{i\in\mathbb{N}}(K_i, B) \in \mathcal{B}(\Gamma^0(\pi))$. Thus, for $A \in \mathcal{B}(\tilde{\Gamma}^0(\pi|_V))$ we can define:

$$\mu'(A) := \mu(\tilde{A} \cap (M, B))$$

where $\tilde{A} := \{\gamma \in \Gamma^0(\pi) : \gamma|_U \in A\}$ which is a set in the Borel algebra as restriction is a continuous operation in the compact-open topology. It is easily shown that $\mu'$ is a measure.

Now for a dense sequence of points $p_i \in V$ define $\tilde{\mu}_n$, a finite sigma-subadditive function on the open subsets of $V^n$ by:

$$\tilde{\mu}_n(A_1 \times \cdots \times A_n) := \mu'\left(\bigcap_{i=1}^{n}(\{p_i\}, A_i)\right)$$

which can be extended to a finite and outer regular measure $\mu_n$ on $V^n$ by $\mu_n(A) := inf\{\tilde{\mu}_n(W) | A \subset W \text{open}\}$ (first, this gives only an outer measure which can then be shown to be a Borel measure by use of Caratheodory's Criterion, cf. [5]).

At this point, one would like to use transitivity of Hamiltonian flows on $V^n$. Unfortunately, this transitivity does not hold on $V^n$ as a whole, as e.g. a point on the diagonal $(v, \cdots, v)$ cannot be mapped outside the diagonal by a flow of the form $H \times H \times \cdots \times H$. Therefore, we consider $V^n$, while itself a manifold, as stratified by partitions of $\{1, \cdots, n\}$ in a natural way: if we have coordinates $v_1, \cdots v_n$, $v_i \in V$, the stratum corresponding to the partition

$\{(i_1^1, \cdots, i_{k(1)}^1), \cdots, (i_1^J, \cdots, i_{k(J)}^J)\}$ (where $J$ is the number of the blocks) is the set:

$$\{(v_1, \cdots, v_n) \in V^n | v_{i_1^1} = \cdots = v_{i_{k(1)}^1}, \cdots, v_{i_1^J} = \cdots = v_{i_{k(J)}^J}\}.$$

Now, we use a fact about Hamiltonian vector fields. A result of Boothby ([1], [8], Theorem 6.4., p. 27) assures that for a symplectic manifold $(S, \omega)$, the compactly supported flows of symplectic vector fields act $\mathbb{N}$-transitive, i.e. for any $2n$-tuple of points in $S$, say $(p_1, \cdots, p_n, q_1, \cdots, q_n)$, with $p_i \neq p_j, q_i \neq q_j$ for $i \neq j$, we can find a symplectomorphic flow mapping $p_i$ to $q_i$ for it all $i$. On simply-connected manifolds these flows are generated by Hamiltonian vector fields (cf. [8], Theorem 6.3, p. 26).

The $\mathbb{N}$-transitivity of Hamiltonian flows $H$ on $V$ implies the transitivity of flows of the form $H \times \cdots \times H$ on every stratum of $V^n$. On every stratum there is a Lebesgue measure invariant as well under these flows. We want to show that on every stratum $\mu_n$ is a multiple of the Lebesgue measure of this stratum. This will be a very consequence from the following:

**Lemma 6.** *Let $V$ be an open subset in $\mathbb{R}^{2n}$ with its standard metric and symplectic form. Let $\nu$ be an outer-regular, finite measure on an open subset $Y = U_1 \times \cdots \times U_k$ of $(\mathbb{R}^{2n})^k$ (with the scalar product $g = n_1 g_{U_1} \oplus n_2 g_{U_2} \oplus \cdots \oplus n_k g_{U_k}$ and the $U_i$ are open subsets of $V$) with $\nu(S) = 0$ for every $(2nk - 1)$-sphere $S$ in $Y$ with radius smaller than a pointwise bound $R$, $\mathcal{L}$ the Lebesgue measure on $Y$, let both $\nu$ and $\mathcal{L}$ be invariant under the transitive action of diffeomorphisms of the form $H \times H \cdots \times H$ where $H$ is a Hamiltonian flow on $V$. Then $\nu$ is a multiple of $\mathcal{L}$.*

**Proof.** Pick an open subset $Y' \subset Y$ with compact closure $\overline{Y'} \subset Y$. First we show that $\mathcal{L}$ is $\nu$-continuous on $Y'$. As $\nu$ is assumed to be outer-regular, the assumption that $\mathcal{L}$ is not $\nu$-continuous implies the existence of a sequence $W_n$ of open subsets with:

$$\rho(W_m) := \frac{\mathcal{L}(W_m)}{\nu(W_m)} > m.$$

Now to each $W_m$ we can apply Vitali's Covering Theorem ([11]). For every $\epsilon > 0$, there is a countable disjoint family of Euclidean balls $B_i$ of radius $< \epsilon$ in $Y$ with $\mathcal{L}(Y \setminus \bigcup_{i \in \mathbb{N}} B_i) = 0$. Now assume that $\rho(A) > L$ and that there is a countable disjoint family $A_n \subset A$ with $\mathcal{L}(A \setminus \bigcup_{i \in \mathbb{N}} A_n) = 0$. If there were no $A_i$ with $\rho(A_i) > L$ we would have

$$\mathcal{L}(A) = \mathcal{L}\left(\bigcup_i A_i\right) = \sum_i \mathcal{L}(A_i) \leq L \sum_i \nu(A_i) = L\nu\left(\bigcup_i A_i\right) \leq L\nu(A)$$

which is in contradiction to the assumption $\rho(A) > L$. Therefore, there is an $A_i$ with $\rho(A_i) > L$.

Now set $B^0(m) := W_m$ and choose with the Vitali step above inductively for any $k \in \mathbb{N}$ balls $B^k$ with radius $< \frac{1}{k}$ which are contained in each other and with $\rho(B^k) > m, \ \forall k \in \mathbb{N}$. Then consider the diagonal sequence $B_n := B^n(n)$. It is a sequence of balls which converge to a point $p \in \overline{Y'} \subset Y$ because $\overline{Y'}$ is compact and as the radii go to zero. Now use the transitivity to get such a sequence of converging balls for every other point $q \neq p$ of $Y$.

Take a map $G \ni H : p \mapsto q$. For every $n \in \mathbb{N}$ consider $H^{-1}(B_{\frac{1}{n}}(q))$ which contains a ball $B_n$ with $\rho(B_n) > n$. Then $H(B_n)$ is in general not a ball itself but contains a second ball $B_n^q$ (via the Vitali step above) which has $\rho(B_n^q) > n$ and for which the Euclidean distance to $q$ satisfies $\mathrm{d}(q, B_n^q) < \frac{1}{n}$.

As neither $\nu$ nor $\mathcal{L}$ feel sufficiently small $(l-1)$-spheres, $\tilde{\rho}(p, r) := \rho(B_r(p))$ is a continuous function in $Y \times [0, 1]$, therefore for every point $q \in Y$ there is a sequence of balls $B'_{r(n)}(q)$ *around* $q$ with $\rho(B'_{r(n)}(q)) \to \infty$.

Now pick some open set $A$ in $Y'$ with $\rho(A) < \infty$. Choose some $K > \rho(A)N$ where $N$ is the constant depending only of the dimension of $Y$ in the Besikovitch lemma below. Then for all $x \in A$ there is an $r(x) > 0$, such that $B_{r(x)}(x) \subset A$ and

$$\mathcal{L}(B_{r(x)}(x)) > K\nu(B_{r(x)}(x))$$

and $A = \bigcup_{x\in A} B_{r(x)}(x)$. The Besikovitch Covering Lemma ([5]) assures that this contains $N = N(dim(Y))$ families of disjoint subcollections which together cover the whole set $A$:

$$A = \bigcup_{i=1}^{N} \dot{\bigcup_{j}} B_{r_j^i}(x_j^i).$$

Thus, there is an $i$ such that:

$$\frac{1}{N}\nu(A) \leq \sum_j \nu(B_{r_j^i}(x_j^i)) \leq K \sum_j \mathcal{L}(B_{r_j^i}(x_j^i)) \leq K\mathcal{L}(A),$$

thus we have $\rho(A) \geq N^{-1}K$ which is a contradiction as we chose $\rho(A)N < K$ in the beginning (recall that $N$ does only depend on the dimension of $Y$!). Therefore $\mathcal{L}$ is continuous with respect to $\nu$.

As $\nu$ is sigma-finite, $\mathcal{L}$ has the form $\mathcal{L}(A) = \int_A f \,\mathrm{d}\nu$ for a measurable function $f : Y' \to [0, \infty]$. We want to use the concept of approximate continuity, cf [5]), to show that $f$ is constant. First note that as $\nu$ is a Radon measure we know that $\nu$-almost everywhere we have:

$$\rho_p(R) := \rho(B_R(p)) = \frac{1}{\nu(B_R(p))} \int_{B_R(p)} f \,\mathrm{d}\nu \to f(p)$$

for $R \to 0$. Pick such a point $p$. Then construct Hamiltonian flows that map $p_i$ to $q_i$ and which are isometric translations in small neighborhoods $U_i \ni p_i$. This gives an element in $G$ which maps $p$ to $q$ and which is an isometry in a small neighborhood of $p$. Therefore at *every* point of $Y$ the limit of $\rho(R)$ is the same. We conclude by the Besikovitch argument above. $\square$

We want to apply the lemma above to a stratum of $V^n$. This is isometric to $V^k$ with $g_{V^k} = n_1 g_V \oplus n_2 g_V \oplus \cdots \oplus n_k g_V$ with $\sum_{i=1}^k n_i = n$. And this in turn is isometric to $n_1 V \times n_2 V \times \cdots \times n_k V$ as subset of Euclidean $\mathbb{R}^{n\cdot k}$. Recall that any open subset of this stratum is an open subset in $n_1 V \times n_2 V \times \cdots \times n_k V$ disjoint from the fat diagonal $\underline{\Delta}$ which consists of all points where two of the coordinates are identical. Hence, it remains to show that:

**Lemma 7.** *Let $\nu$ be a measure on an open subset of $n_1V \times n_2V \times \cdots \times n_kV$ disjoint from the fat diagonal $\underline{\Delta}$ and invariant under the k-fold product of the action of Hamiltonian flows on V. Then sufficiently small spheres in $n_1V \times n_2V \times \cdots \times n_kV$ (as subset of Euclidean $\mathbb{R}^{nk}$) are $\nu$-null sets.*

**Proof.** Given a point $p = (p_1, \cdots, p_k)$, take $R > 0$ with $S_{2R}(p) \cap \underline{\Delta} = \emptyset$, i.e. $B_{2R}(p_i) \cap B_{2R}(p_j) = \emptyset$ for $i \neq j$. Now for every point $s = (s_1, \cdots, s_k)$ of $S_{2R}(p)$ choose a ball with radius $r < \frac{1}{2\sqrt{n}}R$ around $s$. We will consider the pieces $T(s)$ cut out by these balls on the sphere, $T(s) := B_r(s) \cap S_R(p)$ . For each such piece we will construct a Hamiltonian vector field $X_f$ on $V$ for which $X_f^{\times k}$ generates infinitely many disjoint copies of $T(s)$ violating the finite measure condition if we assume that the piece has nonzero measure. Given that it has zero measure we cover the whole sphere with finitely many of these pieces and get that the sphere itself has zero measure. Now let $\rho$ be the radial vector field with respect to $p_j$. As $s \in S_R(p)$, we know that $s_j \in B_R(p_j) \quad \forall j$. Pick $j_0$ with $p_{j_0} \notin B_r(s_{j_0})$. As $r < \frac{1}{2\sqrt{n}}R$, there has to be such a $j_0$. Set the function $f$ to be constructed equal to zero in $V \setminus B_R(p_{j_0})$. We want the Hamiltonian vector field $X_f$ to increase the distance to $p_j$ in the smaller ball $B_r(s_{j_0})$, i.e. to flow outwardly. At the same time, for sake of integrability of the gradient vector field we want the gradient of $f$ to point into radially outward direction as well. The almost complex structure $J$ is at each point a full-rank map of the tangent space without an eigenvector, thus without an invariant $(n-1)$-dimensional subspace. $\rho$ defines at every point $q$ an allowed halfspace $H$ in the tangent space $T_qB$ with a nonempty intersection $H \cap J^{-1}H$ (as otherwise there would be a $(n-1)$-dimensional invariant subspace). As $H \cap J^{-1}H$ depends smoothly on the footpoint, there is a smooth vector field in $B_r(s_{j_0})$ contained pointwisely in this subspace. As $B_r(s_{j_0})$ is simply connected, the vector field (as pointing away from $p_j$ in $B_r(s_{j_0})$) is the gradient of a function $f$ in $B_r(s_{j_0})$ which can be extended arbitrarily, but with support in $B_R(p_{j_0})$, to all of $V$. So we have a function whose Hamiltonian flow $F_t$ increases the distance to $p = (p_1, \cdots, p_k)$ in the chosen neighborhood of $s$. Therefore the images of $T(s)$ under $F_t$ are all disjoint in a small neighborhood, $t \in [0, \epsilon]$:

$$F_{t_1}(s_1) \neq F_{t_2}(s_2)$$

because the opposite would mean that the orbits of $s_1$ and $s_2$ coincide (as $F_t$ is a local diffeomorphism). Thus, we have infinitely many disjoint copies of $T(s)$ with the same measure, therefore $T(s)$ has to be a zero set which implies that the whole sphere does, because of its compactness. $\square$

Now, we apply the lemmata above to every stratum of $V^n$ and to the stratum-wise transitive flows of the form $Fl_{X_f}^{\times i}$ for some $i \in \mathbb{N}$ and some Hamiltonian vector fields $X_f$. If we use the lemmata above for every stratum of $V^n$, we get that the measure $\mu_n$ is of the form:

$$\mu_i = \sum_{j=1}^{i} P_{ij}$$

with $P_{ij}$ being a measure on the $j$-th stratum and of the form:

$$P_{ij}(A) := \sum_{k\in\mathcal{P}^i_j} c_k \mathcal{L}(pr(B_1(k))(A)), \mathcal{L}(pr(B_2(k))(A)), \cdots, \mathcal{L}(pr(B_j(k))(A))$$

where the sum is over partitions $k$ of $\{1, \cdots, i\}$ into $j$ blocks $B_1(k), \cdots, B_j(k)$, $c_k$ being some nonnegative coefficients. The first observation is that for a set $D \subset V$:

$$\mu_i(D^{\times i}) = \sum_i \left( \sum_{k\in\mathcal{P}^i_j} c_k \right) (\mathcal{L}(A))^j \qquad (3)$$

so we define $|P_{ij}| := \sum_{k\in\mathcal{P}^i_j} c_k$. Then we consider the two-dimensional pattern of the $|P_{ij}|$, each being the element in the $i$-th row and $j$-th column:

$$\begin{array}{ccccc}
|P_{11}| & 0 & 0 & 0 & \\
\downarrow{\scriptstyle\geq} & \downarrow & \downarrow & \downarrow & \\
|P_{21}| & |P_{22}| & 0 & 0 & \\
\downarrow{\scriptstyle\geq} & \downarrow & \downarrow & \downarrow & \\
|P_{31}| & |P_{32}| & |P_{33}| & 0 & \cdots \\
\downarrow{\scriptstyle\geq} & \downarrow & \downarrow & \downarrow & \\
|P_{41}| & |P_{42}| & |P_{43}| & |P_{44}| & \\
\downarrow{\scriptstyle\geq} & \downarrow & \downarrow & \downarrow & \\
\cdots & \cdots & \cdots & \cdots & \\
 & \downarrow{\scriptstyle i\to\infty} & \downarrow{\scriptstyle i\to\infty} & \downarrow{\scriptstyle i\to\infty} & \\
 & 0 & 0 & 0 &
\end{array}$$

Then it is obvious that the elements of the first column decrease monotonously with $i$ while the elements in the other columns tend to zero for $i \to \infty$ for fixed $j$ because the values converge to the measure of the set of maps whose image consists of $j$ different points which is empty because $M$ is connected.

We want to show that $|P_{M1}|\mathcal{L}(V) = \mu_M(V^{\times M})$ for all $M \in \mathbb{N}$. The trivial part is $|P_{M1}|\mathcal{L}(V) \leq \mu_M(V^{\times M})$ (cf. Eq. (3)), the nontrivial part is $|P_{M1}| \cdot \mathcal{L}(V) \geq \mu_M(V^{\times M})$. We consider $C \subset V$ with:

$$\frac{\mathcal{L}(V)}{\mathcal{L}(C)} = D > 1.$$

Now given a $\delta > 0$, we take a $k$ such that $|\mu_i(C^{\times i}) - \mu(M, C)| \leq \delta$ for all $i \geq k$ which exists because $\mu_i(C^{\times i}) \to \mu(M, C)$ as this holds for all finite Borel measures on $C^0(U, V)$.

Then assume that there is an $S > i$ that

$$|P_{S1}|\mathcal{L}(C) \leq \mu_S(C^{\times S}) - 4\delta.$$

Then for all $N \geq S$,

$$|P_{N1}|\mathcal{L}(C) \leq \mu_N(C^{\times N}) - 2\delta.$$

Now let w.r.o.g. be $\mathcal{L} < 1$ and set $y(B) := \sum_{i=2}^{\infty}(\mathcal{L}(B))^i$ for a set $B$.
Given an $n \in \mathbb{N}$, choose $Z(n) \in \mathbb{N}$ such that

$$|P_{N2}|, |P_{N3}|, \cdots, |P_{Nn}| < \frac{\delta}{y(C)} \quad \text{for all } N \geq Z(n).$$

This exists because of convergence of the columns in the diagram above. If we insert this estimate into the Eq. (3), we get

$$\mu_N(C^{\times N}) \leq |P_{N1}|\mathcal{L}(C) + \delta + \sum_{i=n+1}^{N} |P_{Ni}|(\mathcal{L}(C))^i$$

and therefore $\sum_{i=n+1}^{N} |P_{Ni}|(\mathcal{L}(C))^i \geq \delta$, and

$$\mu_N(V^{\times n}) \geq \sum_{i=n+1}^{N} |P_{Ni}|(\mathcal{L}(V))^i \geq (\frac{\mathcal{L}(V)}{\mathcal{L}(C)})^{n+1} \sum_{i=n+1}^{N} |P_{Ni}|(\mathcal{L}(C))^i \geq D^n\delta$$

and by choice of a large $n$ we can produce a contradiction as $\mu_N(V^{\times N}) \to \mu((U, V))$ as this holds for all finite Borel measures on $C^0(U, V)$. Therefore, $|P_{M1}| = \frac{\mu((U,V))}{\mathcal{L}(V)}$ for all $M$, i.e. the measure is concentrated on the first column which in the limit represents the constant maps. Therefore $\mu$ is an image measure $\mathcal{P}_*\rho$ of a measure $\rho$ on $V$ under the canonical embedding $\mathcal{P} : V \hookrightarrow C^0(U, V)$ whose image is the set of constant maps (to $\rho$ we could again apply Lemma 6 to prove that it can only be a multiple of the Lebesgue measure). So we have shown that for $C^\infty(U, V)$, at most the measure $\mu = \mathcal{P}_*\rho$ (a measure which only feels the constant maps) can satisfy $X_{\tilde{F}}^*(\mu) = \mu$ for any smooth function $F$ on $U \times V$. Now we choose an arbitrary smooth function $F$ on $U \times V$ *with compact support in* $U \times V$ and consider the Hamiltonian vector field $X_{\tilde{F}}$. Under its flow no constant map will be mapped to a constant map, and we have a contradiction. □

To circumvent the problem described by the proposition above, one could think about several loopholes for constructing appropriate measures: the choice of a polarization, of a linear functional instead of a Borel measure, or of a field theory where the space of initial data is so restricted that it does not allow for an $\mathbb{N}$-transitive symmetry of the measure, which happens very likely in the case of many gauge theories.

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